Arbitrariness in defining fractal basins: Relations between open and closed systems

A. E. Motter[a] *and P. S. Letelier*[b]
Departamento de Matemática Aplicada - IMECC
Universidade Estadual de Campinas, Unicamp
13083-970 Campinas, SP, Brazil

Abstract

A discussion about dependences of the (fractal) basin boundary dimension with the definition of the basins and the *size* of the exits is presented for systems with one or more exits. In particular, it is shown that the dimension is largely independent of the choice of the basins, and decreases with the size of the exits. Considering the limit of small exits, a strong relation between fractals in exit systems and chaos in closed systems is found. The discussion is illustrated by simple examples of one-dimensional maps.

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[a]E-mail: motter@ime.unicamp.br.
[b]E-mail: letelier@ime.unicamp.br.

1 Introduction

Fractals have been largely applied to characterize the dynamics of systems with multiple modes of exit [1]. For instance, in the capture of orbits by attractors [2], in the scattering of particles by located [3] and by nonlocated potentials [4], and in the initial-condition sensitivities caused by chaotic transients [5]. In this context, fractal sets are: Boundaries among basins of initial conditions going out via different exits, singularities of scattering functions, or singularities of exit-time functions.

An exit system has in general either one of the two following kinds of behavior: a) The system is stable, which is the open counterpart of nonchaotic closed systems, or b) the system presents some sensitive dependence on initial conditions, the open analogous of chaotic closed systems. The behavior exhibited by the system is determined by the nature of the nonattracting invariant set of trapped orbits. More precisely, it is through the dimension of the future invariant set that the sensitivity of the system is quantified [1]. This dimension can go from zero (no instabilities) to the energy surface dimension (maximal sensitivity), relating the cause of the sensitivity (invariant sets) with the phenomenon itself (fractal structures in basin boundaries, exit-times, etc).

The aim of this paper is to discuss relations between different fractal sets, the dependence of the dimension of the basin boundaries with the definition of the basins (equivalently, the definition of the exits), and the dynamical consequences of fractals in closed (nonexit) systems. Among the motivations of this work are the so far unanswered questions: (1) How arbitrary can be the definition of the basins? A basin boundary gives information about the system when its dimension reveals the dimension of the future invariant set. Since the future invariant set and the basin boundary are in general non identical sets, conditions on the definition of the exits that ensure the equality of the dimensions have to be found. (2) How to study a system whose orbits "choose" the exits only after the time evolution of the initial conditions by arbitrarily large periods of time? As an example consider the orbits of light traveling around a periodically perturbed black hole [6]. By physical considerations, almost all orbits starting on a Poincaré section are (eventually) captured by the black hole or scattered to infinity. Suppose that we are interested in measuring the sensitivity of the system for these two exits by computing the dimension of the invariant orbits on the Poincaré

section. For some initial conditions, the trajectory integrated on a finite period of time goes unequivocally to the black hole. Other orbits, however, remain around the black hole for a long time and it is in general difficult to determine whether the final state will be at the black hole or at infinity. This problem is a consequence of our inability in integrating the system for an infinite interval of time. (3) What sort of relation communicates fractals in exit systems with chaos in nonexit systems? Fractal structures and chaos present a number of analogous implications. For instance, both chaos and fractals prevent integrability. Accordingly, it worth looking for a common mechanism determining fractals in open systems and chaos in closed systems.

In order to answer these questions we restrict the discussion to a class of autonomous systems for which almost all orbits of interest eventually exit, and the exits are defined regions of the phase space. For these systems, we shall consider each N-dimensional manifestly invariant part of the phase space (e.g., each energy surface), where N is refered as the effective phase space dimension.

In Section 2 we observe that the basin boundary dimension is largely independent of the choices of the basins if the *total* exit is fixed. In this context we also discuss relations between the dimensions of different fractal sets for different concepts of dimension. The dimension decreases, however, when the total exit is enlarged. The enlargement of the exits as a technique to study systems whose capture of orbits by the exits takes arbitrarily long time is considered in Section 3. The limit of small exits, on the other hand, reveals the chaotic or regular character of the corresponding closed system. Relations between fractals in exit systems and chaos in the associated nonexit system are studied in Section 4. Finally, our conclusions are presented in the last section. The discussion is accompanied by explicit examples of one-dimensional maps. Non standard definitions used along the text are summarized in the appendix.

2 Invariances of fractal dimensions

Let us consider the map [1]

$$x_{n+1} = \begin{cases} 2\eta x_n, & \text{if } x_n \leq 1/2, \\ 2\eta(x_n - 1) + 1, & \text{if } x_n > 1/2, \end{cases} \tag{1}$$

where η is a positive parameter ($\eta > 1$ in this section). In spite of its noninversible and discontinuous character, this map can be seen as a canonical model for hyperbolic scattering systems. The noninversibility is a consequence of the reduced dimension of the model. The discontinuity would be avoided by employing a tend-like map or a logistic-like map, which is also smooth, with no relevant changes. We prefer this map, however, because of its simpler symbolic dynamics.

For a uniform distribution of initial conditions in $[0, 1]$, the decay rate with the number n of iterations is η^{-n}. A half of the points goes to $x > 1$ and then asymptotically to $+\infty$, and the other half goes to $x < 0$ and then to $-\infty$. In Fig. 1 we show in two different colors the attraction basins associated to $+\infty$ and $-\infty$ as a function of η. In this figure we see the complex structure of the basin boundary (also present in any magnification), a feature of systems with sensitive dependence on initial conditions. The box-counting dimension of the basin boundary numerically estimated by a method presented in Ref. [1] is fitted by $D_c = \ln 2/\ln 2\eta$. On the other hand, since the points that go out in one iteration are those of the interval $(1/2\eta, 1 - 1/2\eta)$, we can reinterpret the total exit as being this interval. In Fig. 2 we show the exit-time function for $\eta = 3/2$. In computing the uncertainty dimension of the singularities of the exit-time functions by the method of [7] we also obtain a curve fitted by $D_u = \ln 2/\ln 2\eta$.

The coincidence of the above fractal dimensions is due to the presence of an invariant bounded set (the future invariant set) with box-counting dimension $D_c = \ln 2/\ln 2\eta$, which is chaotic since the Lyapunov exponent $h = \ln 2\eta$ is positive. The box-counting dimension is defined by [1]

$$D_c = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln 1/\varepsilon}, \tag{2}$$

where N is the number of ε-interval needed to cover the set. The invariant set is a Cantor set of Lebesgue measure zero obtained as follows. Remove the $1 - 1/\eta$ open interval in the middle of $[0, 1]$. From each one of the two remaining intervals remove the same fraction in their middle, and so on.

In terms of Fig. 1, the invariant points correspond to the nonisolated points of the basin boundary for each η. A countable set of isolated points, like $x = 1/2$, are in the basin boundary but are not invariant. This set, refered here as the *intrinsic frontier*, is a consequence of the definition of the basins and, in this case, do not affect the dimension of the boundary. (The

intrinsic frontier is completely avoided by coloring in black and white the points that leave $[0,1]$ in even and odd number of iterations, respectively - see Fig. 3.) Moreover, the basin boundary dimension remains invariant if we redefine the exits as $(1/2\eta, \lambda]$ and $(\lambda, 1-1/2\eta)$ for $1/2\eta < \lambda < 1-1/2\eta$ (Fig. 1 corresponds to the choice $\lambda = 1/2$). In Fig. 4 we show the basin boundary as a function of λ for $\eta = 3/2$. In spite of the morphological changes in the basins, the variation of λ just dislocates the intrinsic frontier so that the basin boundary dimension is still the dimension of the invariant set.

Let us put the above discussion in a more general context, where we consider two exits E_1 and E_2 so that $E_1 \cap E_2 = \emptyset$, $E_1 \cup E_2 = (1/2\eta, 1-1/2\eta)$, $\mu(E_1) \neq 0$ and $\mu(E_2) \neq 0$ (μ denotes the Lebesgue measure). The question we address is: How arbitrary can be the choices (definitions) of the exits E_1 and E_2 ? We denote by L the set of extremes of the intervals that appear in the construction of the Cantor set, by I_0 the remaining invariant set, by I the (total) invariant set[1] $L \cup I_0$, by F_1 the set of points of the intrinsic frontier in $(1/2\eta, 1-1/2\eta)$ (that go out in the first iteration), by F_i points of the intrinsic frontier that go out in the i^{th} iteration, by F the (total) intrinsic frontier $\cup_{i=1}^{\infty} F_i$, and by B the (total) basin boundary $F \cup I$.

Taking the Hausdorff dimension (see [1] for the definition),

$$\begin{aligned} D_H(B) &= \max\{D_H(F), D_H(I_0), D_H(L)\} \\ &= \max\{D_H(F_1), D_H(I_0)\}, \end{aligned} \tag{3}$$

where we used $D_H(L) = 0$ since L is a countable set, and $D_H(F) = D_H(F_1)$ because F is a countable union of similar copies of F_1 [8]. Therefore, the dimension of the basin boundary is equal to the dimension of the restricted invariant set if and only if $D_H(F_1) \leq D_H(I_0)$. When this inequality is not satisfied, the basin boundary dimension is just an effect of the definitions of the exits. In the sense of Hausdorff dimension, are good exits those ones for which $D_H(F_1) = 0$. It includes all F_1 countable, as in the case of E_1 and E_2 defined as finite unions of intervals.

Alas, in general we cannot measure D_H but D_c, where by definition $D_c(A) \geq D_H(A)$ for any set A [1]. Considering the box-counting dimension:

$$D_c(B) = \max\{D_c(F), D_c(I_0), D_c(L)\}. \tag{4}$$

[1]We call I_0 the *restricted* invariant set. The distinction between I_0 and I is in order to show that there is no relevant differences between an open total exit $(1/2\eta, 1-1/2\eta)$ with invariant set I and a closed total exit $[1/2\eta, 1-1/2\eta]$ with invariant set I_0.

From (2) we can show that $D_c(L) = D_c(I_0)$ and $D_c(F) = \max\{D_c(F_1), D_c(I_0)\}$. If F_1 is finite, $D_c(F_1) = 0$ and $D_c(B) = D_c(I_0)$. It is the case for the exits consisting of a finite union of intervals. One example is provided by Fig. 4, where $F_1 = \{\lambda\}$. If F_1 is infinite, however, $D_c(F_1)$ may be positive even if F_1 is countable (see [1], p. 103). The equality, $D_c(B) = D_c(I_0)$, is satisfied whenever $D_c(F_1) \leq D_c(I_0)$. In this case we also have $D_H(B) = D_c(B) = D_H(I_0) = D_c(I_0)$. The basin boundary dimension is greater than the invariant set dimension when $D_c(F_1) > D_c(I_0)$.

In numerical computation, the box-counting dimension is estimated from the uncertainty dimension (D_u) [1, 7]. It is conjectured that typically $D_u = D_c$ in dynamical systems (see [7] and references therein). Let us argue about this identity in our example. First we consider the invariant set I. A point x_0 is ε-uncertain if $(x_0 - \varepsilon, x_0 + \varepsilon) \cap I \neq \emptyset$, and we denote by $f_I(\varepsilon)$ the fraction of points that are ε-uncertain. The dimension of the invariant set is given by

$$D_u(I) = 1 - \lim_{\varepsilon \to 0} \frac{\ln f_I(\varepsilon)}{\ln \varepsilon} \ , \tag{5}$$

and it follows from (2) that $D_u(I) = D_c(I)$. Now we consider the boundary set B for F_1 finite. Introducing the exits E_1 and E_2, a point x_0 is ε-uncertain if $x_0 - \varepsilon$ and $x_0 + \varepsilon$ go out by different exits. The fraction of uncertain points is

$$\begin{aligned} f_B(\varepsilon) &= \alpha f_I(\varepsilon) + f_F(\varepsilon) \\ &= \alpha f_I(\varepsilon) + \beta f_I(\varepsilon), \end{aligned} \tag{6}$$

where α and β are constants. Here, $(1-\alpha) f_I$ is the fraction of points x_0 in f_I such that $x_0 - \varepsilon$ and $x_0 + \varepsilon$ are in the same basin, and it implies the term $\alpha f_I(\varepsilon)$. The contribution of the intrinsic frontier is $f_F(\varepsilon)$, which represents the scale structure of I for F_1. It implies the term $\beta f_I(\varepsilon)$. Accordingly,

$$\begin{aligned} D_u(B) &= 1 - \lim_{\varepsilon \to 0} \frac{\ln f_B(\varepsilon)}{\ln \varepsilon} \\ &= 1 - \lim_{\varepsilon \to 0} \frac{\ln f_I(\varepsilon)}{\ln \varepsilon} \\ &= D_c(I) \ \ (= D_c(I_0)). \end{aligned} \tag{7}$$

This identity allows the use of the uncertainty dimension, which has nice numerical properties, to calculate the box-counting dimension.

Back to Fig. 2, the set of singularities (S) of the exit-time function (set of point where the exit-time is infinite) corresponds to the invariant set I. For the exit-time T, a point x_0 is ε-uncertain if $T(x_0-\varepsilon) \neq T(x_0+\varepsilon)$. Therefore, in the computation of the uncertainty dimension we are in fact estimating the box-counting dimension of the discontinuities of the exit-time function. (In this case there is no intrinsic frontier, since the exit-time is defined with respect to the total exit.) The set of discontinuities can be identified with L, forming sequences that converge to the singularities. It implies

$$D_u(S) = D_c(L) = D_c(I). \tag{8}$$

Once more the uncertainty dimension is the same as the box-counting dimension.

Although discussed for a specific case, the above results are far more general since our system models a vast class of exit systems. For any system with a finite number of basins, the dimension of the union of all basin boundaries is equal to the dimension of at least one of the boundaries. So, it is enough to consider systems with only one boundary (two exits, E_1 and E_2.) When the effects of the intrinsic frontier are avoided, the basin boundary has the same dimension as the future invariant set. Since the latter set does not depend on E_1 or E_2 but just on $E_T = E_1 \cup E_2$, it leads to an invariance of the dimension with the definition of the basins. Roughly speaking, in spite of possible morphological changes, the basin boundary dimension will be the same for any nonpathological couple of exits E_1' and E_2' with positive probability of being reached, even since satisfying $E_T = E_1' \cup E_2'$. Then, there is usually great freedom in defining the exits for a fixed total exit. This invariance of fractal dimension allows us to choose among several alternatives the best one to measure the invariant set dimension and hence the sensitivity of the system. Also, it gives us confidence that the result reflect a true dynamical property of the system and not an immaterial effect.

In this section we studied measurements of fractal dimensions when the total exit is fixed and, therefore, the invariant set is fixed. Next, in Sections 3 and 4, we discuss the complementary situation, that is, variations of the invariant set (and of its dimension) under changes of the total exit.

3 Dependences with the total exit

A fundamental feature of map (1) is the monotonous increase of the (future) invariant set dimension with the reduction of the parameter η. For $\eta > 1$, the invariant set is a zero measure set satisfying $D(I) \to 1$ for $\eta \to 1$. This property is very general and closely related to a conjecture of Moura and Letelier [4] in the context of Hamiltonian systems: The dimension of the invariant set of a chaotic scattering system goes to the effective phase space dimension when the energy tends to the escape energy. In both cases the reduction of a parameter, η or the energy, blinds the exits and increases the number of oscillations of typical orbits before going out. Accordingly, it "fattens" the invariant set, increasing its dimension.

Let us consider an extension of the above situation to one where the total exit is enlarged. Instead of $(1/2\eta, 1 - 1/2\eta)$ (the natural exit determined by points that go to $\pm\infty$) we define the new total exit $(1/2\eta - \alpha, 1 - 1/2\eta + \alpha)$ for $0 < \alpha < 1/2\eta$ and $\eta > 1$. The invariant set $I(\eta, \alpha)$ will be in $[0, 1/2\eta - \alpha] \cup [1 - 1/2\eta + \alpha, 1]$ and its dimension can be easily computed from the exit-time function. In Fig. 5 we show the dimension of the invariant set as a function of α for $\eta = 1.1$. The fractal dimension is a decreasing function of α, but *not* strictly decreasing. The dimension presents a pattern of plateaux due to the existence of gaps in the invariant set. A gap $(1/2\eta - \alpha_2, 1/2\eta - \alpha_1) \cup (1 - 1/2\eta + \alpha_1, 1 - 1/2\eta + \alpha_2)$ in the invariant set $I(\eta, \alpha_1)$ implies a plateau in the interval (α_1, α_2): In increasing α the dimension changes only when the enlarged exit advances on points of the invariant set $I(\eta, \alpha)$. The result is a graph with a staircase structure where the derivative is null in the complement of a Cantor set. Moreover, the limit of small α is well defined and recover the original dimension, that is, $D(I(\eta, \alpha)) \to D(I(\eta, 0)) = \ln 2/\ln 2\eta$ for $\alpha \to 0$.

It is instructive to consider the invariant set explicitly in terms of its symbolic dynamics. The symbolic dynamics can be associated to the invariant set of map (1) as follows. For the exit $(1/2\eta, 1 - 1/2\eta)$, an invariant point $x_0 \in [0, 1]$ is represented by $0.\epsilon_0\epsilon_1 ... \epsilon_n ...$, where $\epsilon_n = 0$ if $M^n(x_0) \leq 1/2\eta$ and $\epsilon_n = 1$ if $M^n(x_0) \geq 1 - 1/2\eta$. The grammar of this dynamics is the Bernoulli shift grammar and $M^n(x_0)$ is represented by $0.\epsilon_n\epsilon_{n+1} ...$. To simplify, suppose the exit is enlarged only on the left becoming $(1/2\eta - \beta, 1 - 1/2\eta)$, where β is so that points in $(1/2\eta - \beta, 1/2\eta)$ are represented by $0.011...1\epsilon_m\epsilon_{m+1}...$ for some integer m. The points of the invariant set with respect to the exit

$(1/2\eta, 1-1/2\eta)$ that go out in fewer than $k \times m$ iterations are those points represented by sequences of 0 and 1 with the word $0\underbrace{11...1}_{m-1}$ somewhere in the first $k \times m$ letters. Subtracting these points, the fraction of invariant points with respect to $(1/2\eta, 1-1/2\eta)$ that remain invariant with respect to the exit $(1/2\eta-\beta, 1-1/2\eta)$ for $k \times n$ iterations is

$$N(k,m) = 2^{km}\left\{1-\sum_{s=1}^{k}\left[\frac{(-1)^{s+1}}{2^{sm}}\sum_{i_s}^{(k-s)n+1} \dots \sum_{i_2}^{i_3}\sum_{i_1}^{i_2} 1\right]\right\}. \tag{9}$$

It gives the scale of the hierarchical construction of the Cantor set structure, from which follows that the box-counting dimension of the new invariant set is

$$D_c(I(\eta,\beta)) = \lim_{k\to\infty}\frac{\ln N(k,m)}{\ln(2\eta)^{km}}\ . \tag{10}$$

For $m=3$, the computation of the sequences in a different order leads to

$$N(k,3) = 2^{3k-1}\left\{\sum_{j=2}^{3k} a_j + \sum_{j=3}^{3k} b_j\right\}, \tag{11}$$

where $a_2 = b_3 = 1$,

$$a_j = -\frac{f_{j-2}}{2f_j}\sum_{i=2}^{j-1} a_i \text{ for } j\geq 3,\ b_j = -\frac{f_{j-1}-1}{2f_{j+1}-2}\sum_{i=3}^{j-1} b_i \text{ for } j \geq 4, \tag{12}$$

and $\{f_n\}$ are the Fibonacci numbers. The case $m=2$ is trivial since $N(k,2) = 2k+1$ and the dimension of the invariant set is zero. The dimension computed from (9-12) is identical to that one computed by the uncertainty method. In Fig. 6 we show this fractal dimension, normalized by the factor $\ln 2/\ln 2\eta$, as a function of m. The exit size is reduced as m is augmented, increasing the dimension.

The enlargement of exits is useful and sometimes necessary in numerical computation of fractal dimensions. An example is given by the orbits of light traveling around a black hole, as discussed in Section 1. The difficulty in determining the final state (at the black hole or at infinity) is removed by enlarging the exit at infinity. The new exit may be defined as $r > r_0$, instead of $r=\infty$, where r is an adequate distance from the attractor center.

The dimension so obtained is an under bound of the exact invariant set dimension (associated to the original exits) since part of the invariant set is lost throughout the new exit.

The difficulty illustrated here is present in a vast class of systems. It has frequently appeared in the literature where, in general, some kind of enlargement of the exits is implicitly taken into account. The above discussion intends to provide subsidies to clarify the effects of this procedure.

4 Transitions from exit to nonexit systems

Exits can be defined even when the system has no natural exits. As an example consider map (1) once more, but now for $\eta = 1$ (it is analogous for $1/2 < \eta < 1$). In such case the interval [0,1] is mapped to itself, and the symmetrical exit defined in Section 3 reduces to $(1/2 - \alpha, 1/2 + \alpha)$. In Fig. 7 we show a graph of the dimension of the (zero measure) invariant set as a function of α. The graph presents a pattern of plateaux similar to that of Fig. 5. In fact, it is a consequence of a simple relation existing between fractal dimensions for $\eta > 1$ and for $\eta = 1$. This relation follows from a one-to-one correspondence, via symbolic dynamics[2], between invariant sets for different values of η. In terms of symbolic dynamics, the exit $(1/2\eta - \alpha, 1 - 1/2\eta + \alpha)$ can be denoted by $E = (0.0\epsilon_1\epsilon_2... \ , \ 0.1\epsilon_1'\epsilon_2'...)$, where $\epsilon_i, \epsilon_i' \in \{0,1\}$ and $\epsilon_i + \epsilon_i' = 1$. This notation ignores the plateaux due to the original exit since $0.0\epsilon_1\epsilon_2... = \psi_\eta(\alpha)$ where the function ψ_η is constant on gaps of the original invariant set. Accordingly, from (2) follows that the box-counting dimensions for $\eta = 1$ and $\eta > 1$ differ just by the overall scale factor $\ln 2 / \ln 2\eta$,

$$D_c(I(\eta, E)) = \frac{\ln 2}{\ln 2\eta} D_c(I(1, E)). \tag{13}$$

(Relation (13) is valid for any exit, not only for the one considered here, e.g. Fig. 6 where $d = D_c$ for $\eta = 1$.)

An important feature exhibited in Fig. 7 is that the dimension is fractal and goes to 1 as α goes to 0. This behavior is closely related to the fact that the original map, with no exit, is chaotic: the map has a dense set of

[2]For $\eta = 1$, all the interval $[0, 1]$ is associated to the Bernoulli shift symbolic dynamics. The picture is analogous to that one for $\eta > 1$, except that the original invariant set (for $\alpha = 0$) has measure zero for $\eta > 1$ and measure one for $\eta = 1$.

unstable periodic orbits and the Lyapunov exponent is positive ($h = \ln 2$). In order to better understand this relation, let us consider the logistic map $x_{n+1} = rx_n(1 - x_n)$ in the unit interval $[0, 1]$ for $r = 3.8$ (chaotic) and for $r = 2.8$ (nonchaotic). In Fig. 8 we plot the exit-time function for $r = 3.8$ with respect to the exits $(0.61, 0.69)$ (Fig. 8a) and $(0.63, 0.67)$ (Fig. 8b). These graphics present successive steps in the construction of the Cantor structure of the invariant set. They must be compared with Fig. 9, where we show exit-times for $r = 2.8$. The exit-time for $r = 3.8$ presents a complex structure that becomes even more complex when the exit is reduced. It reflects the increasing of the invariant set dimension, $D_u = 0.62$ in Fig. 8a, $D_u = 0.88$ in Fig. 8b, and $D_u \to 1$ in the limit of small exit. A different behavior takes place when $r = 2.8$. In fact, Figs. 9a and 9b show a regular exit-time where no complexity is added when the exit is reduced. In terms of invariant set dimension it means that its value is zero even in the limit of small exit. The transition of the dimension of the invariant set from small exit to no exit is, in this case, discontinuous since it jumps from 0 to 1.

The behavior described in preceding paragraphs is very general and may be summarized as follows: Chaotic nonexit systems present invariant sets with *fractal* dimension when small exits are defined[3], and the invariant set dimension tends to the effective phase space dimension when the exits are arbitrarily reduced. On the other hand, nonchaotic nonexit systems do not present any fractal structure in the invariant set when exits are introduced, and the invariant set dimension jumps (discontinuously) to the effective phase space dimension when the exits are removed[4]. We conjecture that this behavior is typical for dynamical systems in general.

Basically, this conjecture states that chaos in closed systems and fractals in open systems are different manifestations of the same phenomenon. Chaos in closed systems is essentially determined by the presence of a somewhere dense set of unstable periodic orbits. Fractal structures in exit systems are determined by a fractal invariant set that consists of unstable periodic orbits and an uncountable number of nonperiodic orbits surrounding them. Therefore, the introduction and removal of exits leads from one situation to

[3]The exits have to be sufficiently small in order to avoid the complete outcome of the invariant set.

[4]There are pathological examples of nonchaotic systems which exhibit fractal properties when exits are created (see [9]). It happens when the invariant set of the exit system is fractal but nonchaotic. This behavior is, however, atypical.

the other (In most cases, because of the density property, the box-counting dimension of the periodic orbits equals the future invariant set dimension in exit systems and the effective phase space dimension in closed systems.) Physically, an exit system with fractal invariant set evolves chaotically for a period of time before been scattered. When the exits are removed the system evolves chaotically forever. Conversely, chaotic nonexit systems present sensitive dependence on initial conditions. The introduction of small exits detect this sensitivity through singularities on exit-times, scattering functions, etc.

It is also opportune to consider a couple of related works, [5] and [10]. In the former, the authors studied the Sinai billiard with two holes in the external wall. They show that the boundary between the initial-condition basins of attraction defined by the two exits is fractal. In the context of our discussion, this result is a consequence of the well known chaotic dynamics of the (closed) Sinai billiard. In addition, we verified numerically that the basin boundary dimension in fact tends to the effective phase space dimension when the two exits are reduced. In the latter work, the question over whether or not the Bianchi IX cosmological model is chaotic was finally solved. The trouble with this question was that standard indicators of chaos, like Lyapunov exponents, are not invariant under space-*time* diffeomorphisms and cannot be used since the model is relativistic. Following a suggestion first made by [11], the authors show that the system is chaotic by defining three exits and measuring the basin boundary dimension. (The approximation of the dimension to the effective phase space dimension when the exits are reduced was also verified.) In that work, the relation between exit and nonexit systems together with the invariant character of fractal methods played a fundamental role in the solution of the problem.

The potential use of our conjecture as an invariant method to study chaos in closed systems of several dimensions is considered in [12].

5 Conclusions

We have studied, via examples, a number of questions concerning fractals in exit systems. In particular the three questions stated in Section 1 were discussed in Sections 2, 3 and 4, respectively. We observed that basin boundary dimensions are independent of the definitions of the exits since the total exit is fixed and the intrinsic frontier effects are avoided. The dimensions in-

crease, however, with the reduction of the total exit. Our main result refers to the limit of small exits, where a distinctive behavior was found concerning whether the corresponding closed system is chaotic or not. In the former the dimension changes continuously, while in the latter it jumps avoiding non-integer values. Special attention was also given to understand the relations between the dimensions of different fractal sets, and the relations between different concepts of dimension.

Finally, we stress that most of the results presented here are also valid for nonhyperbolic systems [7]. It is the case of the invariances of fractal dimensions discussed in Section 2, since the arbitrariness in choosing the exits is analogous in hyperbolic and nonhyperbolic systems. Concerning Section 3, however, the dimension changes with the total exit only when the nonhyperbolic character is lost in enlarging the exit. In such case, the behavior of hyperbolic systems discussed in Section 3 may take place. Nonhyperbolic exit systems are supposed to have invariant set with maximal dimension (equal to the dimension of the effective phase space) [7]. In this sense, and in the context of Section 4, the relation between closed and open nonhyperbolic systems is even stronger than in the hyperbolic analogous.

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APPENDIX

The following definition refers to phase spaces of autonomous systems for both maps (discrete time) and vector fields (continuous time): (1) An *exit* E is any set of the phase space (possibly at infinity) which may be reached in a finite time or asymptotically. (2) The *total exit* E_T is the union of all exits of the system. (3) The *size* of an exit is given by its phase space volume, and an exit E is *smaller* than another exit E' if E is a proper subset of E'. The size of the total exit is defined analogously. (4) Given a set of exits $E_1, E_2, ..., E_n$, the *basin of attraction* A_i of the exit E_i is the set of all initial conditions whose time evolution reaches E_i before reaching any other exit[5]. (5) The *basin boundary* $B_{i_1,...,i_k}$ is the set of points arbitrarily close to points of A_{i_1},..., $A_{i_{k-1}}$, and A_{i_k}. (6) A *scattering function* is a function

[5] A_i would be defined as the closure of this set, with no relevant changes.

that describes the behavior of the final phase space variables (when the exits are reached) as a function of the initial ones. (7) The *exit-time function* ("time delay" in scattering processes) is a function which associates to the initial conditions the minimal time needed to reach the total exit. (8) The *nonattracting invariant set* is a set composed by nonattracting orbits that do not reach any exit (not even asymptotically) for both directions of the time. Its *stable* and *unstable manifolds*, when exist, are the future and past invariant sets, respectively.

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Figure 1: Portrait of the basins of map (1) as a function η. The initial conditions were chosen on a grid of 400×400. Regions in black and white correspond to orbits that escape to $+\infty$ and $-\infty$, respectively.

Figure 2: Exit-time of map (1) for $\eta = 3/2$ and total exit $(1/2\eta, 1 - 1/2\eta)$.

Figure 3: Portrait of the basins of map (1) as a function η. The initial conditions were chosen on a grid of 400×400. Regions in black and white correspond to orbits that escape from $[0, 1]$ in an even and an odd number of iterations, respectively.

Figure 4: Portrait of the basins for map (1) as a function of λ for $\eta = 3/2$. The initial conditions were chosen on a grid of 800×800. Regions in black and white correspond to orbits that escape to $(1/2\eta, \lambda]$ and $(\lambda, 1 - 1/2\eta)$, respectively.

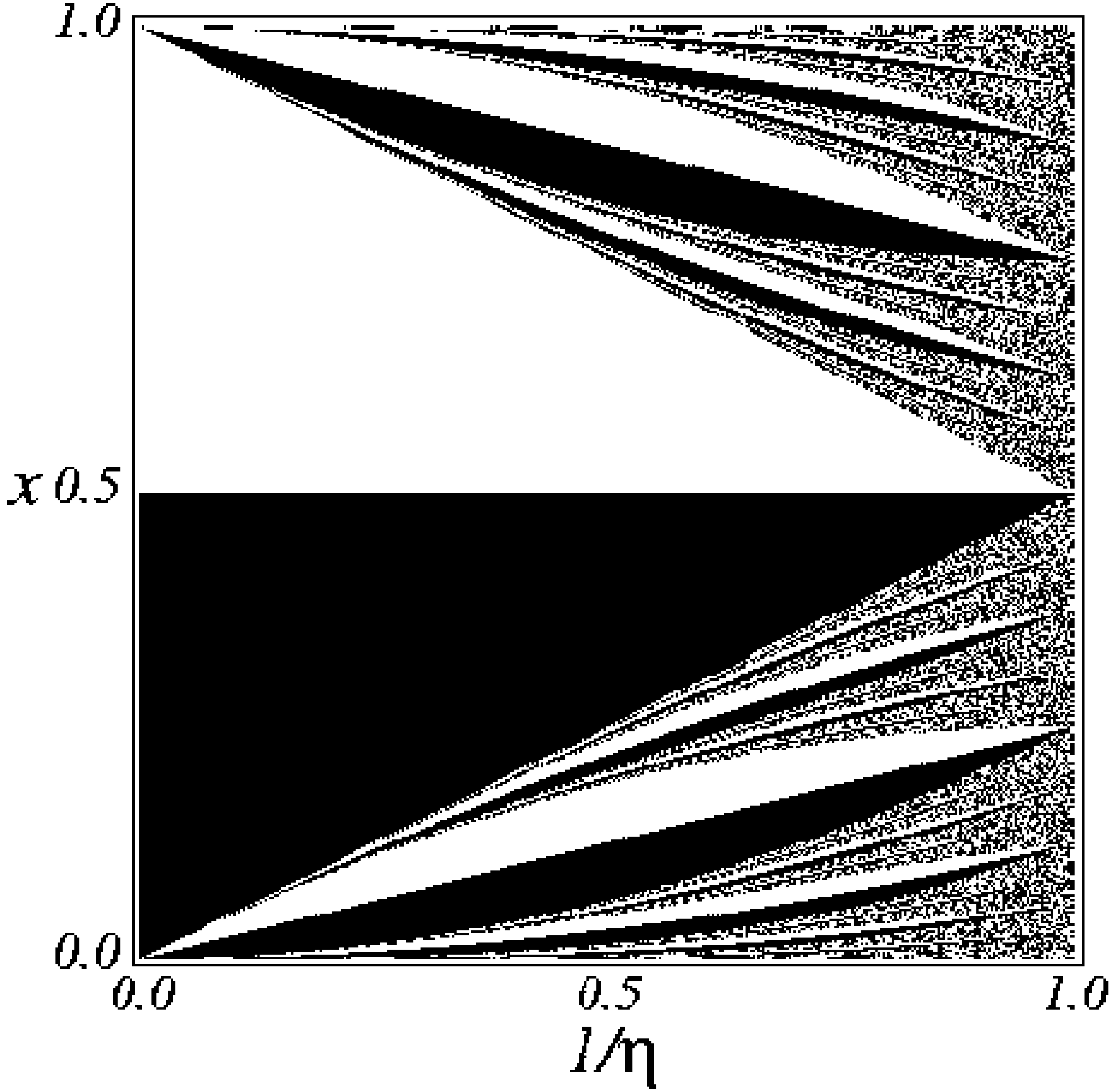
Figure 5: Uncertainty dimension of the invariant set of map (1) for $\eta = 1.1$ and total exit $(1/2\eta - \alpha, 1 - 1/2\eta + \alpha)$ as a function of α.

Figure 6: The normalized box-counting dimension $d = D_c \ln 2\eta / \ln 2$ of the invariant set of map (1) for the total exit $(1/2\eta - \beta, 1 - 1/2\eta)$ as a function of m.

Figure 7: Uncertainty dimension of the invariant set of map (1) for $\eta = 1$ and total exit $(1/2 - \alpha, 1/2 + \alpha)$ as a function of α.

Figure 8: Exit-time of the logistic map for $r = 3.8$ with total exit: (a) $(0.61, 0.69)$; (b) $(0.63, 0.67)$.

Figure 9: Exit-time of the logistic map for $r = 2.8$ with total exit: (a) $(0.61, 0.69)$; (b) $(0.63, 0.67)$.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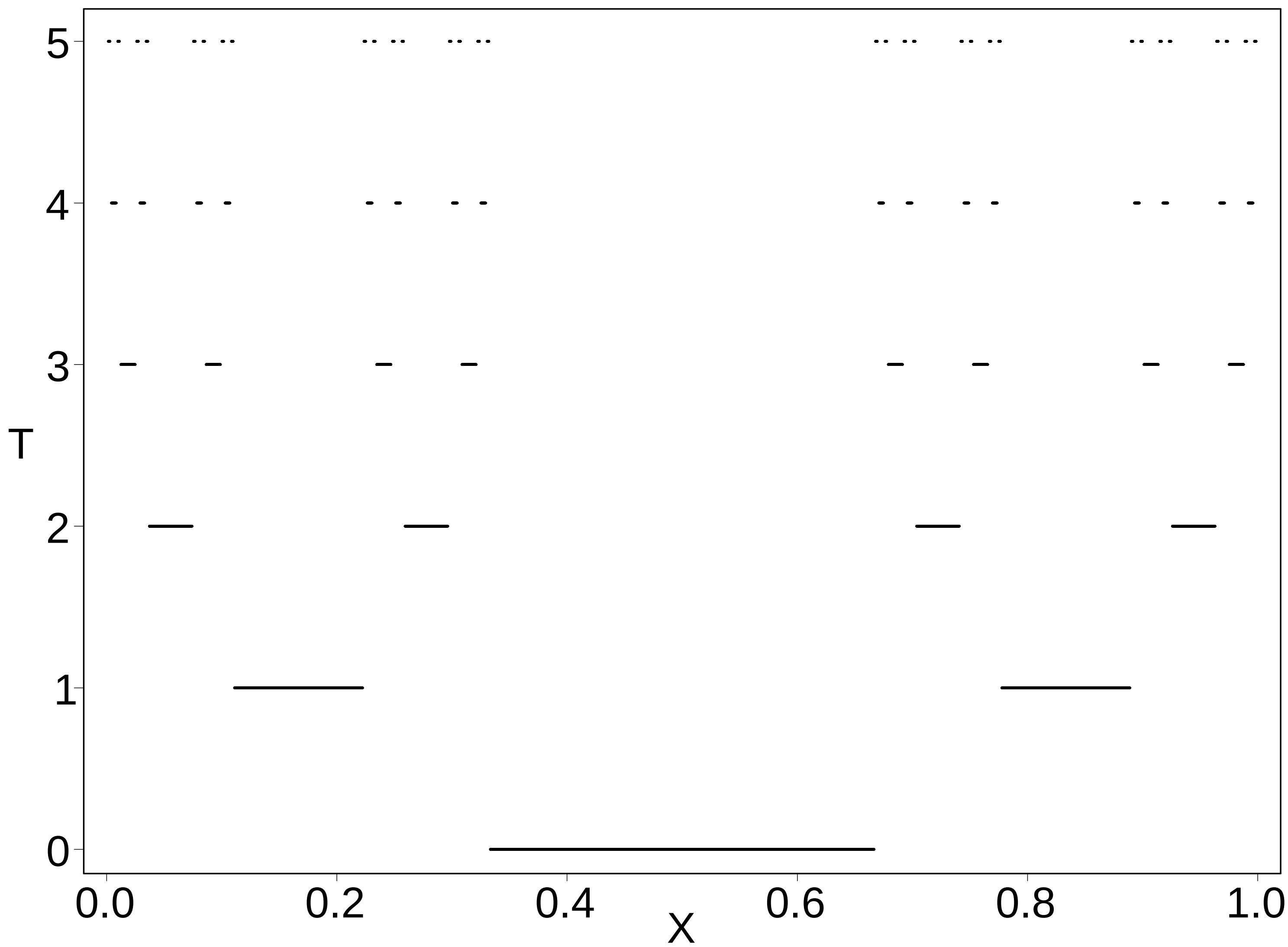
x 0.5
0.0
0.0
0.5
1.0
1/η


5
4
3
2
1
0
T
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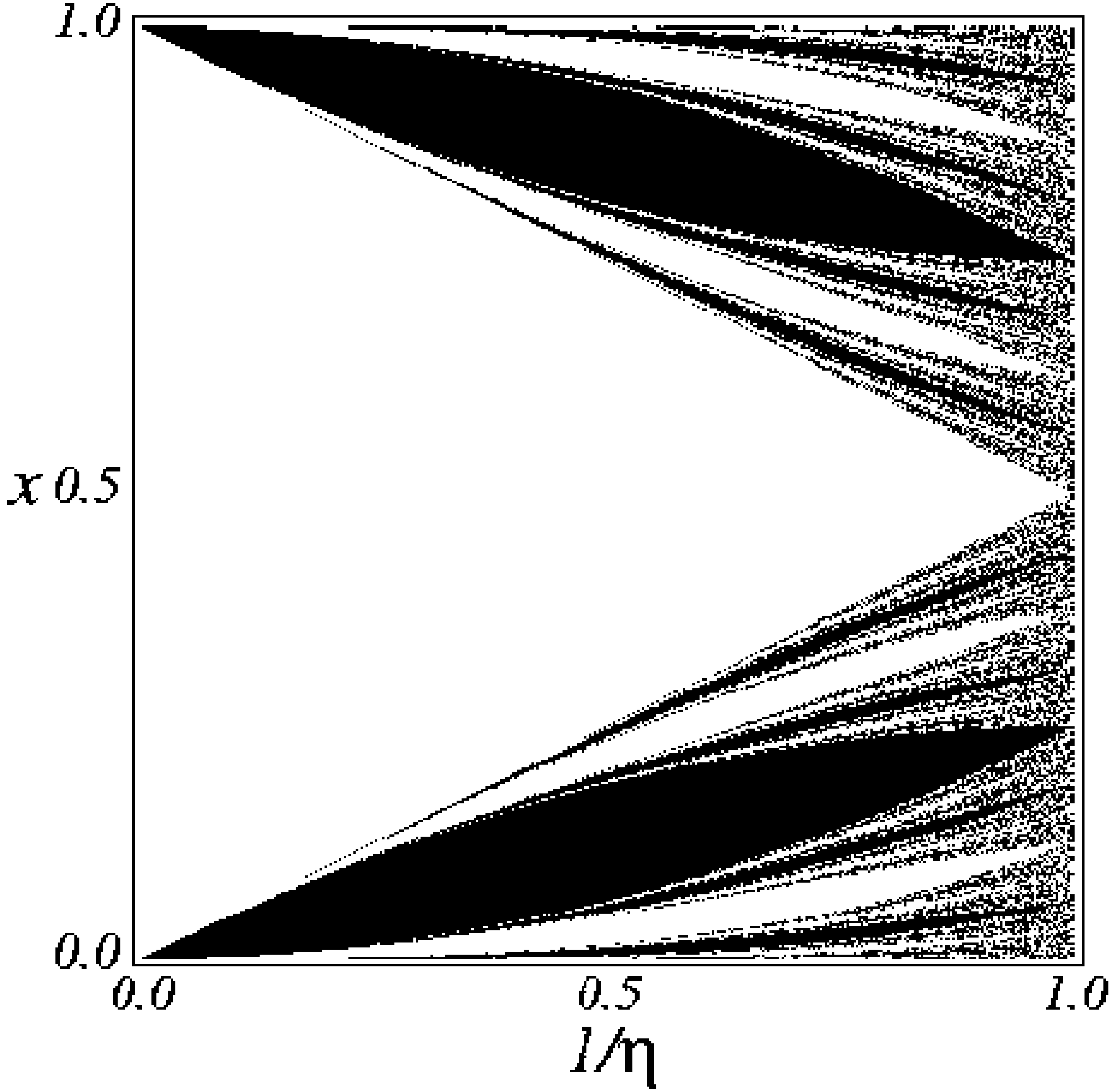
0.2
0.4
0.6
0.8
1.0
X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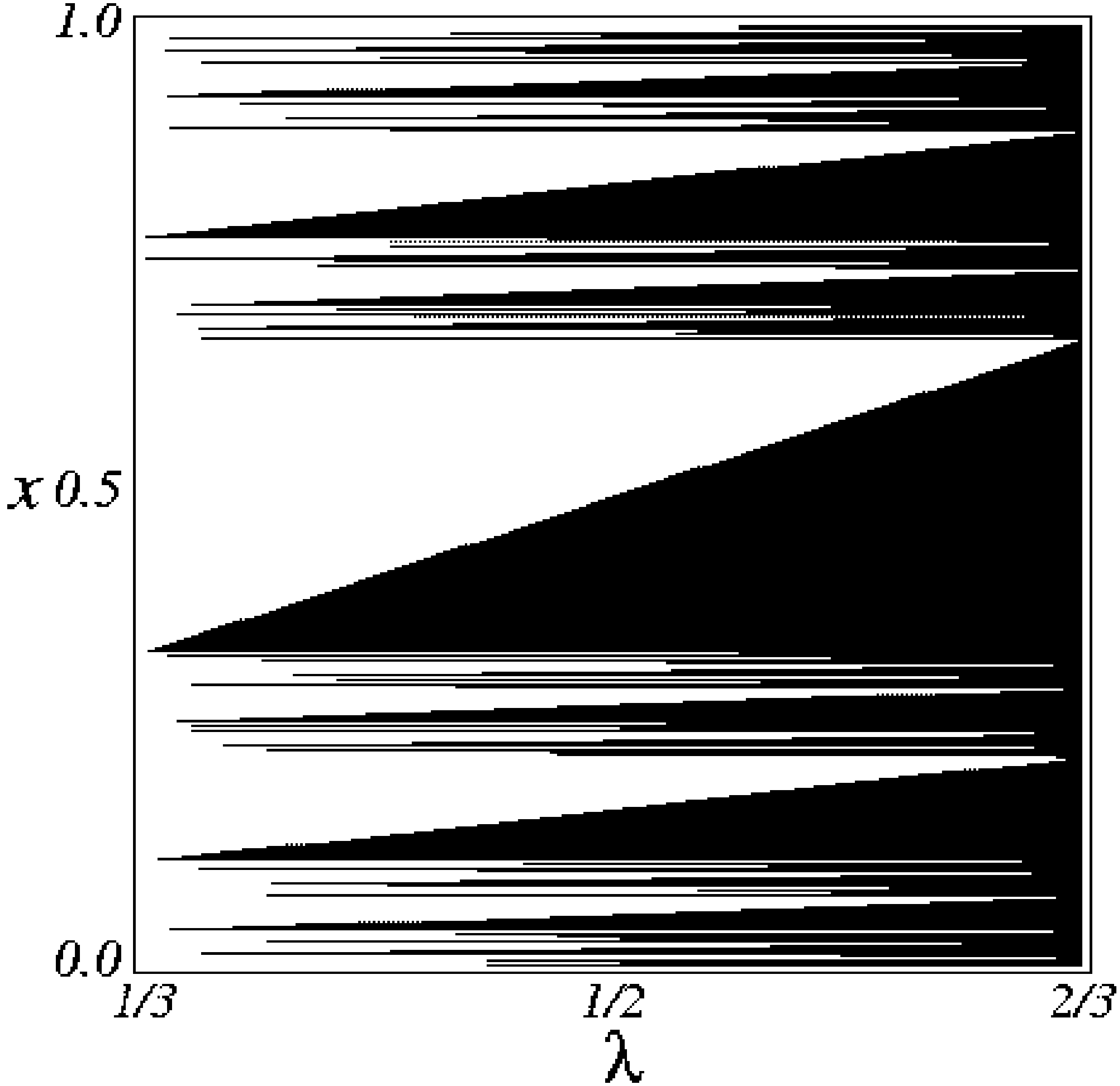
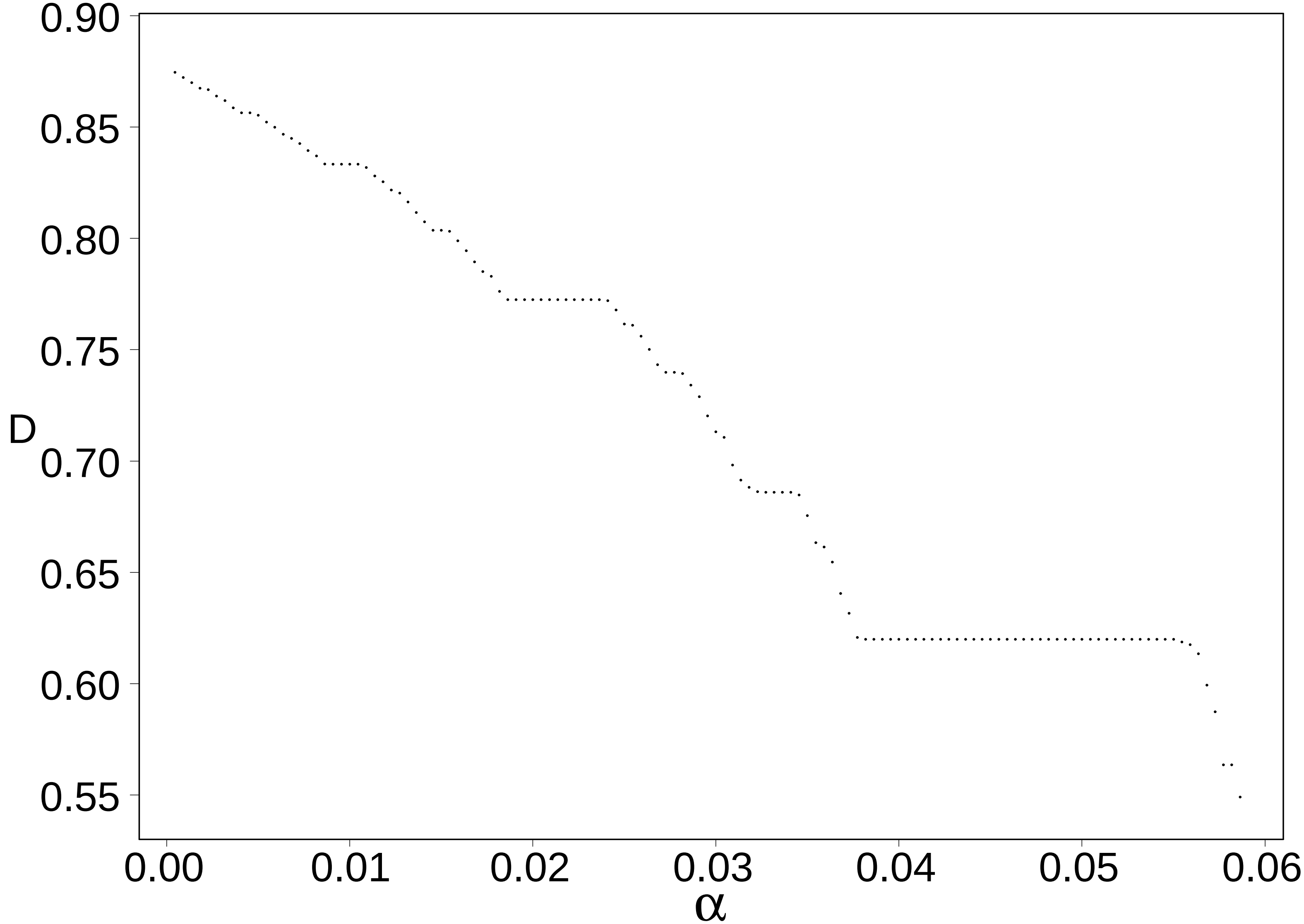
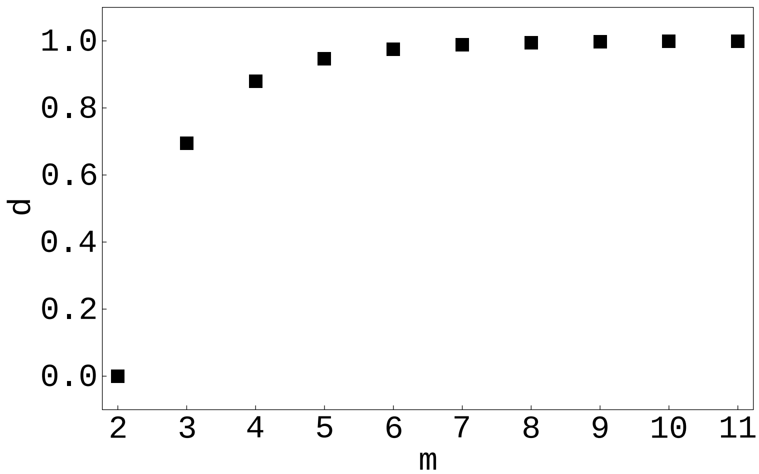
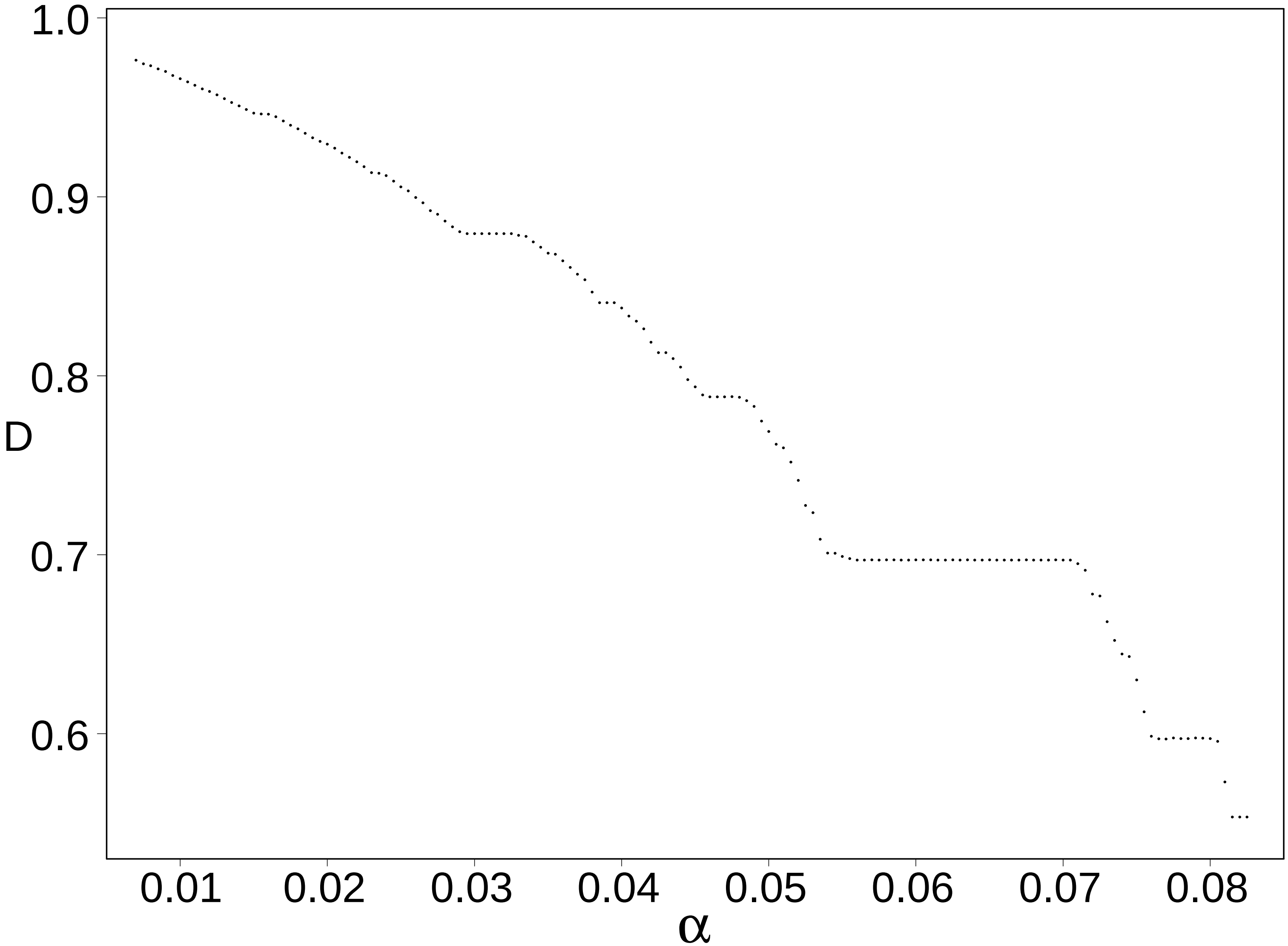
1/η


1.0
x 0.5
0.0
1/3
1/2
2/3
λ

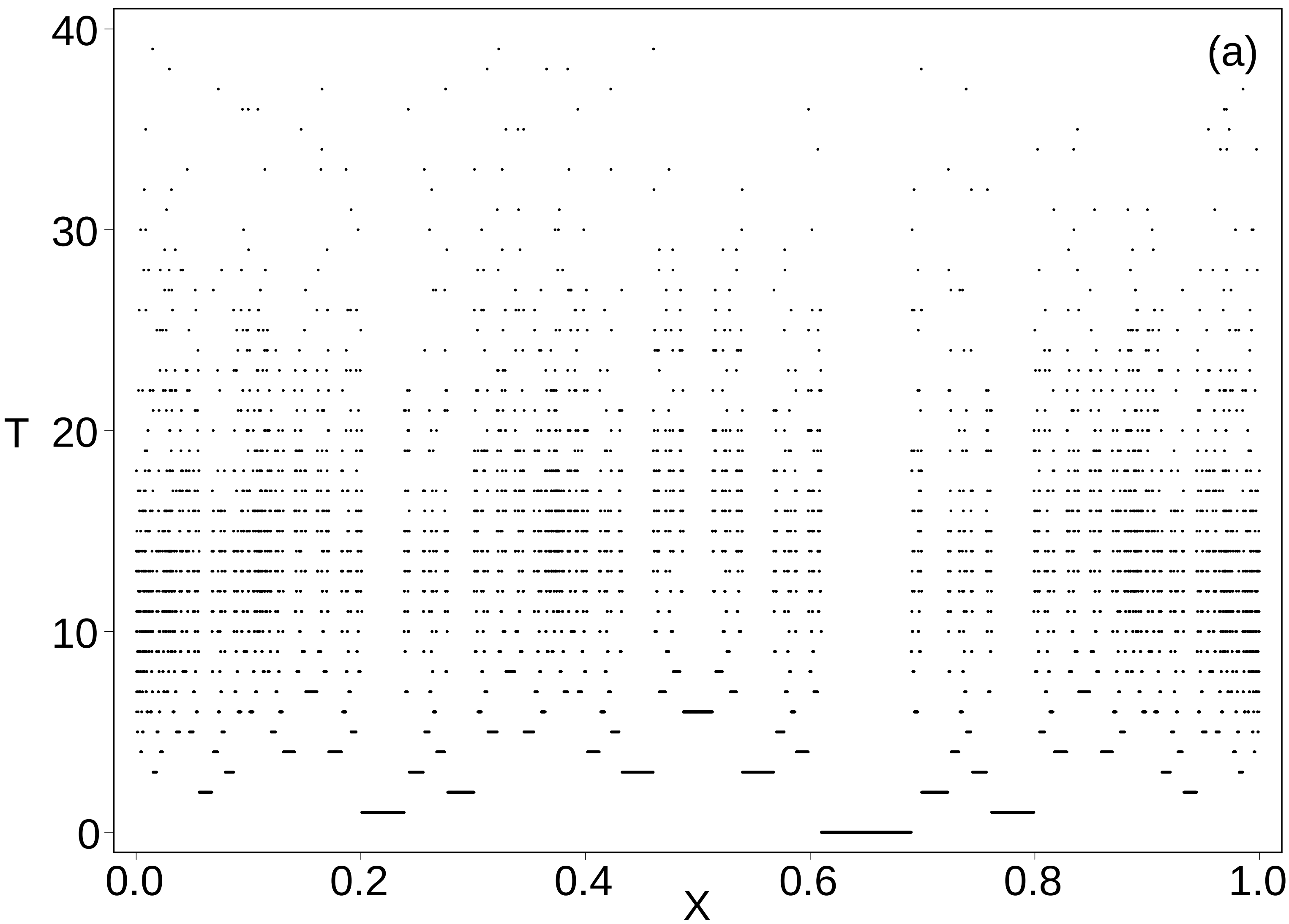

0.90
0.85
0.80
0.75
0.70
0.65
0.60
0.55
D
0.00
0.01
0.02
0.03
0.04
0.05
0.06
α



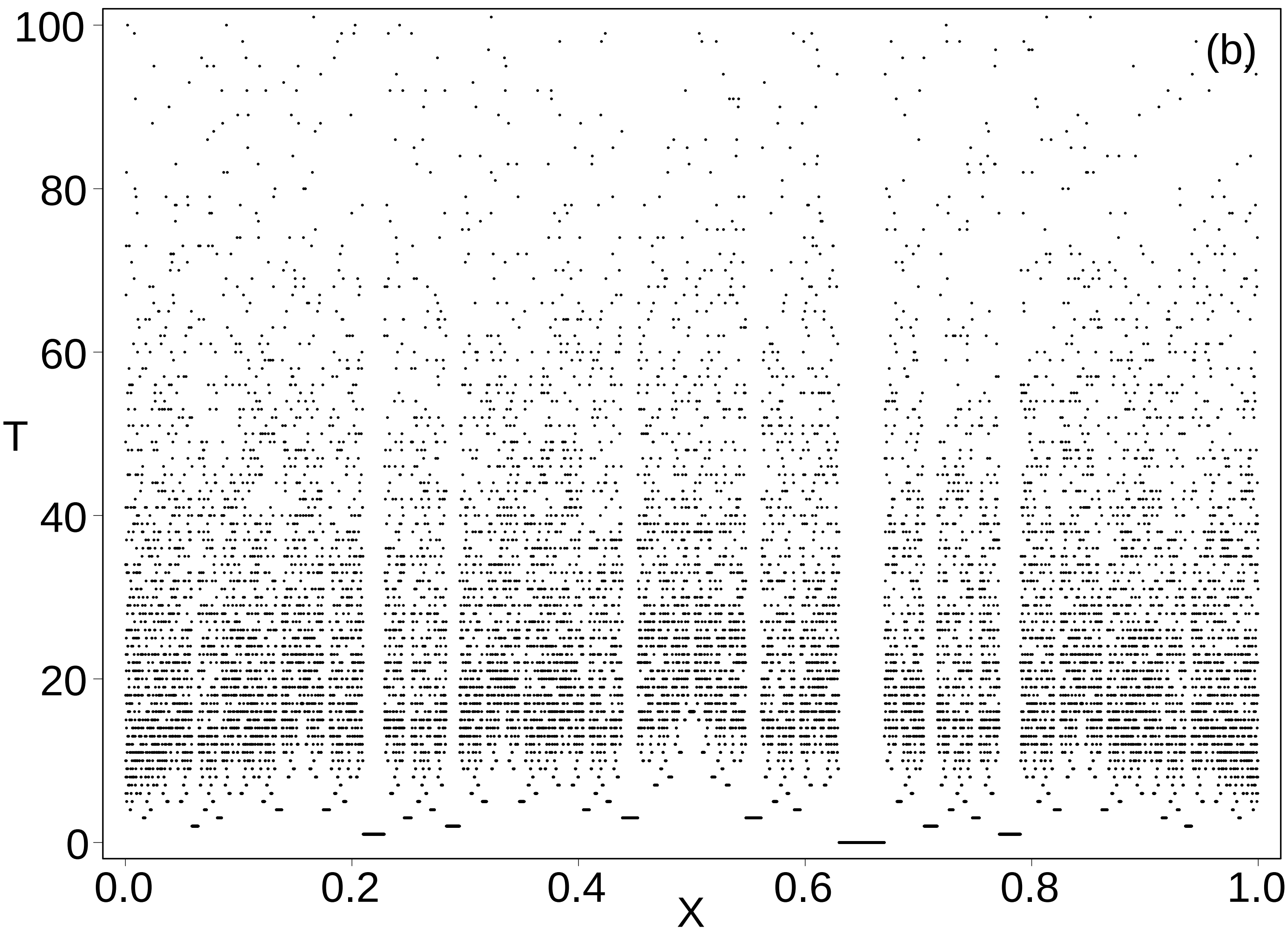
1.0
0.8
0.6
0.4
0.2
0.0
d
2
3
4
5
6
7
8
9
10
11
m



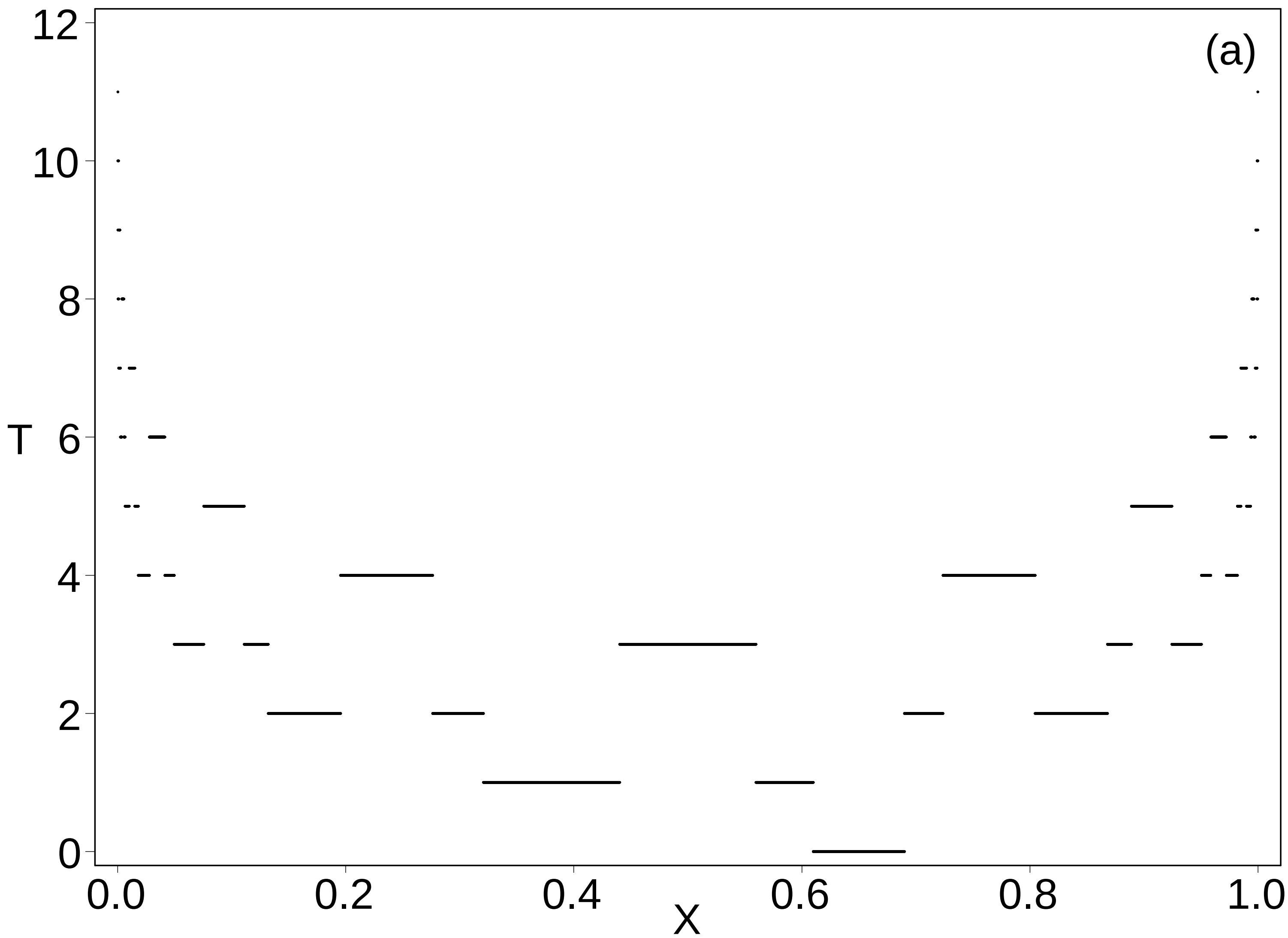
1.0
0.9
0.8
0.7
0.6
D
0.01
0.02
0.03
0.04
0.05
0.06
0.07
0.08
α

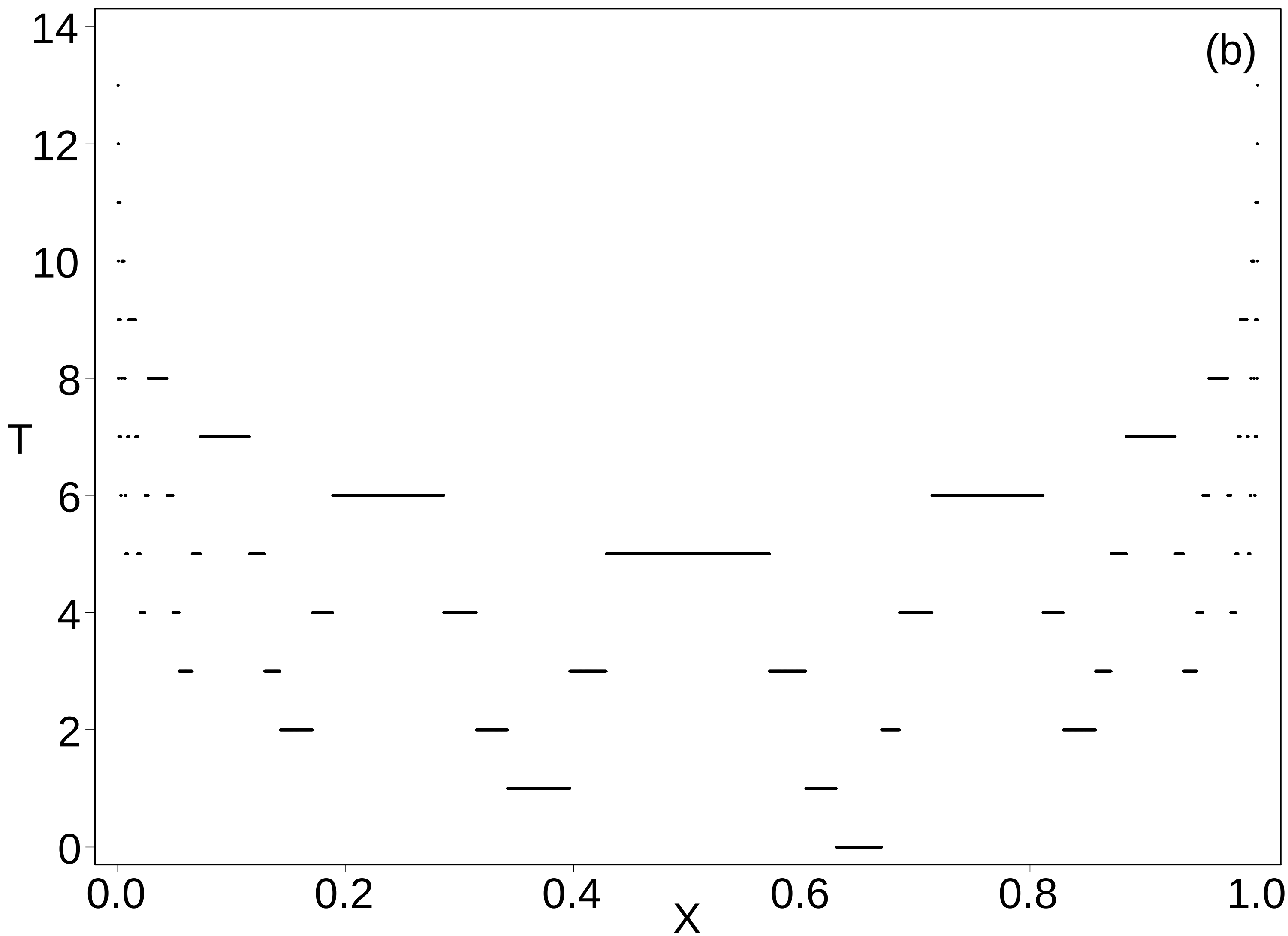


(a)
T
40
30
20
10
0
0.0
0.2
0.4
0.6
0.8
1.0
X



(b)
T
100
80
60
40
20
0
0.0
0.2
0.4
0.6
0.8
1.0
X


(a)
T
0
2
4
6
8
10
12
X
0.0
0.2
0.4
0.6
0.8
1.0



(b)
T
X
0
2
4
6
8
10
12
14
0.0
0.2
0.4
0.6
0.8
1.0